The coordinate of this point is determined from (26), and is independent of the time. In particular, for a sphere

$$\frac{\bar{r}}{b}=\left[\left(\frac{3g+n+6}{n+3}\right)\left(\frac{\lambda^{(n+3)/n}-1}{\lambda^{(3g+n+6)/n}-1}\right)\right]^{n/(3g+3)}(1<n-g<2); \tag{32}$$

and for a cylinder

$$\frac{\bar{r}}{b}=\left[\frac{(g+2)(\lambda^{2/n}-1)}{\lambda^{(2g+4)/n}-1}\right]^{n/(2g+2)}(1<n-g<2). \tag{33}$$

It is seen that the coordinates (32) and (33) differ insignificantly from the corresponding coordinates for intersection of the elastic and steady distribution diagrams, and for $\eta\to\infty$ agree exactly with the coordinates for intersection of the elastic stress intensity distribution with the ideal plastic distribution. In combination with (30) and (31), this result affords a possibility of involving an electronic computer (or using it minimally) to compute the stress–strain state of high-pressure vessels by means of (8) and (9) even in the case $\beta\neq 1$ as an approximate estimate during design. The lower bound of the fracture time is determined from (29) or from the expression $t_*\geq \overline{t_*^0}$ proposed in [5]. In combination with (30), (31) and (29), the relationships (8) and (9) yield the exact solution for $\beta=1$.

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ST. VENANT PRINCIPLE FOR STRONGLY ANISOTROPIC ELASTIC MEDIA

Yu. A. Bogan

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The presence of strong anisotropy in modern composites (consequently, large parameters are present in the generalized Hooke's law for the average stresses) results in limit models being characterized by the phenomenon of "propagation" of the stress state [1].

In this connection, the question occurs as to what degree does the St. Venant principle remain valid for media with inextensible fibers? As is shown below, exponentiality decreasing the potential strain energy with distance from the domain of self-equilibrated load application occurs [2] for media with inextensible fibers under definite conditions; however, it is hence generally impossible to make a deduction about the exponentiality of the damping with distance from the loaded section.

Therefore, the St. Venant principle must be formulated in a weakened, integral form without local estimates of the stress state of the structure for the application of the principle to media with inextensible fibers.

1. Without pinpointing any specific model of a linearly elastic composite, let us take the generalized Hooke's law relationship in the form

$$\sigma_\xi = A_{11}\varepsilon_\xi + A_{12}\varepsilon_\eta,\ \ \sigma_\eta = A_{12}\varepsilon_\xi + A_{22}\varepsilon_\eta,\ \ \tau_{\xi\eta} = G\gamma_{\xi\eta}, \tag{1.1}$$

where $\xi = x\cos\alpha = y\sin\alpha$; $\eta = -x\sin\alpha + y\cos\alpha$; $0\leq\alpha<\pi$ is some constant angle, and (x, y) are cartesian orthogonal coordinates. Let us put

$$\varepsilon^{-2} = A_{11}G^{-1},\ d_{12} = A_{12}G^{-1},\ d = A_{22}G^{-1},$$
$$\bar{\sigma}_\xi = \sigma_\xi G^{-1},\ \bar{\sigma}_\eta = \sigma_\eta G^{-1},\ \bar{\tau}_{\xi\eta} = \tau_{\xi\eta} G^{-1}$$

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and let us retain the previous notation for the dimensionless stresses henceforth. Let us assume that $\varepsilon \ll 1$. This case corresponds to a single directed composite with very stiff fibers parallel to the ξ axis. Passing to the limit in (1.1) as $\varepsilon \to +0$, we obtain the governing relationships

$$\sigma_\xi = q_1 + d_{12}\varepsilon_\eta, \quad \sigma_\eta = d\varepsilon_\eta, \quad \tau_{\xi\eta} = \gamma_{\xi\eta}, \quad \varepsilon_\xi = 0 \tag{1.2}$$

in which (1.2) $q_1 = q_1(\xi, \eta)$ is the Lagrange multiplier corresponding to the kinematic constraint of inextensibility along the ξ axis. For $\varepsilon \neq 0$ the equation for the stress function $w(\xi, \eta)$ has the following form in the absence of volume forces

$$d\varepsilon^2\partial^4 w/\partial\eta^4 + (d - \varepsilon^2 c)\partial^4 w/\partial\xi^2\partial\eta^2 + \partial^4 w/\partial\xi^4 = 0, \tag{1.3}$$

where $c = d_{12}^2 + 2d_{12}$. In the limit as $\varepsilon \to +0$ equation (1.3) goes over into the equation

$$d^{-1}\partial^4 w/\partial\xi^4 + \partial^4 w/\partial\xi^2\partial\eta^2 = 0. \tag{1.4}$$

In constrast to (1.3), equation (1.4) is not already elliptic but composite [3] with a double family of real characteristics $\eta \equiv$ const.

2. Let us consider the question of the influence of strong anisotropy on the "rate" of stress attenuation in an orthotropic half-strip. By using numerical analysis it was shown by some examples in [4] that for a quite stiff armature parallel to the long sides, the stresses attenuate considerably more slowly with distance from the end face than in the isotropic case. The problem of a half-strip is of interest in connection with the following circumstances: It is a model for the St. Venant principle; the solution of the problem of a half-strip is the boundary-layer component in the asymptotic of the first boundary-value problem of elasticity theory for a rectangular domain when one of its measurements is small [5], and therefore permits clarification of the question about the influence of the domain size on the stress attenuation "rate" in the boundary layer under strong anisotropy of the material. The answer to the last question is of practical value. In fact, if the average mechanical characteristics of a unidirectional composite must be measured, then specimens must be selected sufficiently long in order to eliminate the influence of the slowly attenuating boundary layer.

Let $Q = \{(x, y), |y| \leq h, 0 \leq x < +\infty\}$ be an elastic half-strip, and let the orthotropy axes be parallel to the sides of the half-strip. For $x = 0$ and $y = \pm h$ let us pose the following boundary conditions:

$$\sigma_x|_{x=0} = p_1(y), \quad \tau_{xy}|_{x=0} = p_2(y), \quad \sigma_y|_{y=\pm h} = 0, \quad \tau_{xy}|_{y=\pm h} = 0. \tag{2.1}$$

Under the assumption that the load is self-equilibrated and the strain potential energy is finite [6], the boundary-value problem (1.3) and (2.1) has a unique solution which decreases exponentially at infinity. The solution of the boundary-value problem (1.3) and (2.1) is represented in the form [7]

$$w(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n x} w_n(y),$$

where $w_n(y)$ are eigenfunctions and λ_n are the eigennumbers of the following spectral problem:

$$d\varepsilon^2 \frac{d^4 w_n}{dy^4} + \lambda_n^2 (d - \varepsilon^2 c) \frac{d^2 w_n}{dy^2} + \lambda_n^4 w_n = 0; \tag{2.2}$$

$$w_n(\pm h) = 0, \quad \frac{dw_n}{dy}(\pm h) = 0. \tag{2.3}$$

The spectral problem (2.2) and (2.3) is singularly perturbed (a small parameter enters as a factor in the highest derivative). The asymptotic behavior of spectral problems of this kind as $\varepsilon \to +0$ has been studied well [8]. It follows from the results of [8] that a boundary-layer phenomenon occurs near the boundary $y = \pm h$ since the boundary condition $dw_n/dy(\pm h) = 0$ is not satisfied in the limit as $\varepsilon \to +0$. The shortened problem has the form

$$d\frac{d^2 w_n}{dy^2} + \lambda_n^2 w_n = 0, \quad w_n(\pm h) = 0.$$

The eigennumbers have an asymptotic of the form $\lambda_n(\varepsilon) \sim \sum_{k=0}^{\infty} \lambda_n^{(k)} \varepsilon^k$, where $\lambda_n^{(0)} = 0$ or an eigennumber of the shortened problem. Appropriate eigenfunctions have an asymptotic of the form

$$w_n(y, \varepsilon) = \varepsilon G_1^{(n)}(y, \lambda_n(\varepsilon), \varepsilon) \exp[-\varepsilon^{-1}\lambda_n(\varepsilon)(h-y)] + \varepsilon G_2^{(n)}(y, \lambda_n(\varepsilon), \varepsilon) \exp[-\varepsilon^{-1}\lambda_n(\varepsilon)(h+y)] + z_n(y, \varepsilon)$$

and in the limit $\lim_{\varepsilon\to+0} w_n(y, \varepsilon) = z_n(y, 0)$ or zero, where $z_n(y, 0)$ is an eigenfunction of the shortened problem. Near the ends of the range $(-h, h)$ the convergence is not uniform. In this case the eigennumbers and eigenfunctions of the shortened problem are determined explicitly as:

$$\lambda_n = \pm \frac{n\pi}{h} d^{1/2}, \quad z_n(y, 0) = \sqrt{2} \sin \frac{n\pi y}{h}.$$

Therefore, the following deduction can be made from the preceding discussion: As $\varepsilon \to +0$ the solution of the boundary-value problem (1.3), (2.1) loses the property of exponentiality of the attenuation. In fact, for a half-strip bounded by inextensible filaments parallel to the x axis, we obtain the following representation of the solution in the limit as $\varepsilon \to +0$:

$$w(x, y) = \sum_{n=1}^{\infty} \left[a_n + \frac{hd^{-1/2}}{n\pi}(1 - \exp(-n\pi x d^{1/2} h^{-1}))\, b_n\right] \sin \frac{n\pi y}{h} \tag{2.4}$$

(a_n and b_n are determined in terms of the values σ_x and τ_{xy} for $x = 0$) from which it follows that $w(x, y)$ will attenuate exponentially at infinity only for $\sigma_x(0, y) = 0$. The circumstance that although the limit solution (2.4) does not itself attenuate exponentially at infinity, but the estimate

$$E(z) \leqslant E(0) \exp[-2kz], \tag{2.5}$$

where

$$2E(z) = \iint_{Q_z} (d^{-1}\sigma_y^2 + \tau_{xy}^2)\, dxdy; \quad Q_z = \{(x, y) \in Q, \quad x > z\}.$$

holds for the strain potential energy, is of interest. In fact,

$$2E(z) = \sum_{n=1}^{\infty} \frac{n\pi}{h} d^{-1/2} b_n^2 \exp[-2n\pi z d^{1/2} h^{-1}].$$

Since $\exp[-2n\pi z d^{1/2} h^{-1}] \le \exp[-2\pi z d^{1/2} h^{-1}]$ for $n \ge 1$, we hence obtain the estimate (2.5) with the constant k equal to $\pi d^{1/2} h^{-1}$.

3. Now, let us investigate the influence of strong anisotropy on the stress attenuation "rate" in the boundary layer in the following two cases: a) The rectangular domain is extended along the x axis; b) the rectangular domain is stretched along the y axis but the quite stiff armature is parallel to the x axis. The asymptotic expansion of the first boundary-value problem of elasticity theory for a quite long orthotropic rectangle is constructed in [5].

Let us first examine case a). Let $Q = \{(x, y),\ 0 \leqslant x \leqslant a,\ |y| \leqslant h\},\ \gamma = h/a \ll 1$. In this case, as is shown in [5], the boundary-layer phenomenon occurs for small γ near the sides $x = 0, a$. Here functions of boundary-layer type are solutions of the problem (1.3), (2.1) for an elastic half-strip. For small γ the boundary-layer-type solution has the representation $w = \sum_n w_n(\eta) \exp(-\lambda_n t)$, where $t = x/a\gamma$; $\eta = y/h$; $w_n(\eta)$ and λ_n are determined from the solution of the spectral problem (2.3), (2.4). As has been shown above, the spectral problem (2.3), (2.4) is singularly perturbed relative to the parameter ε. Therefore, exponentials of two kinds are present in a boundary-layer-type solution in $\exp\left[-\frac{\lambda_n^{(1)}\varepsilon x}{a\gamma}\right]$ and $\exp\left[-\frac{\lambda_n^{(0)} x}{a\gamma}\right]$, since either $\lambda_n(\varepsilon) = \lambda_n^{(0)} + \varepsilon\lambda_n^{(1)} + \ldots$, or $\lambda_n(\varepsilon) = \varepsilon \lambda_n^{(1)} + \ldots$. The presence of an exponential of the first kind shows that the "rate" of stress decrease in the boundary layer in γ depends substantially on the ratio ε/γ, i.e., on the mutual influence of the domain size and the degree of material anisotropy. For instance, for $\varepsilon = o(\gamma^2)$ the stresses in the boundary layer will attenuate very slowly.

Let us consider case b). Let $Q = \{(x, y),\ |x| \le h,\ 0 \le y \le a\}$. It can be shown directly that the exponentiality of the boundary-layer attenuation in γ is conserved as $\varepsilon \to +0$. Indeed, in this case the eigenvalues of the spectral problem

$$\frac{d^4w_n}{d\eta^4}+\lambda_n^2(d-\varepsilon^2c)\frac{d^2w_n}{d\eta^2}+d\varepsilon^2\lambda_n^4w_n=0, \tag{3.1}$$

$$w_n(\pm 1)=0, \frac{dw_n}{d\eta}(\pm 1)=0, \quad \eta=x/h$$

are determined from the system of equations [5]

$$v\sin\lambda_n v\cos\lambda_n u-u\sin\lambda_n u\cos\lambda_n v=0,$$

$$v\cos\lambda_n v\sin\lambda_n u-u\cos\lambda_n u\sin\lambda_n v=0,$$

where

$$u^2=(d-\varepsilon^2c+D)/2;\ v^2=(d-\varepsilon^2c-D)/2;\ D=[(d-\varepsilon^2c)^2-4d\varepsilon^2)^{1/2}.$$

For small $\varepsilon>0$, $u^2\sim d-\varepsilon^2(1+c)+o(\varepsilon^4)$, $v^2\sim\varepsilon^2+o(\varepsilon^4)$. Passing to the limit in (3.2) as $\varepsilon\to+0$, we obtain the relations $\lambda_n\sqrt{d}=0$, $\tan\lambda_n\sqrt{d}=\lambda_n\sqrt{d}$, to determine λ_n, from which it follows that for $\varepsilon=0$ the spectral problem (3.1) has two sets of negative eigenvalues.

Therefore, the qualitative behavior of solutions of boundary layer type are substantially distinct for small γ in these two cases.

4. It must be noted that if the characteristics of the limit equations are not parallel to the half-strip boundary, then exponentiality of the stress attenuation occurs far from the loaded side. For instance, let Q be an elastic half-strip bonded by two families of inextensible fibers at the angles $\pm\pi/4$ to the x axis. Upon passing to the limit in (1.4) as $d\to+\infty$, and replacing the coordinates (ξ,η), $\xi=2^{-1/2}(x+y)$, $\eta=2^{-1/2}(y-x)$ by the coordinates x, y, the equation for the stress function acquires the form

$$\partial^4w/\partial x^4-2\partial^4w/\partial x^2\partial y^2+\partial^4w/\partial y^4=0. \tag{4.1}$$

Since the boundary is not a characteristic, the boundary conditions are taken in the form (2.1). As in the case of an anisotropic medium, under the assumption of self-equilibration of the load on the side $x=0$, the equation (4.1) under the boundary conditions (2.1) has the unique solution

$$w=\sum_{n=0}^{\infty}\exp(-\lambda_n x)\,w_n(y),$$

where λ_n are the eigennumbers, and w_n are the eigen- and associated functions of the following spectral problem

$$\frac{d^4w_n}{dy^4}-2\lambda_n^2\frac{d^2w_n}{dy^2}+\lambda_n^4w_n=0,\ w_n(\pm h)=0,\ \frac{dw_n}{dy}(\pm h)=0.$$

To determine the eigennumbers, we obtain the equation

$$(2\lambda_n h)^2-\mathrm{sh}^2(2\lambda_n h)=0. \tag{4.2}$$

Upon making the change of variable $z=i2\lambda_n h$, $i=\sqrt{-1}$, it goes over into the equation $z^2-\sin^2 z=0$ which is encountered in solving the first boundary-value problem for an isotropic strip [9]. The roots of this latter equation are complex, and the asymptotic of the roots of large absolute value for the equation $z^2-\sin^2 z=0$ is given by the relationship

$$z\sim\pm\frac{2t+1}{4}\pi\pm i\ln\left(\frac{2t+1}{2}\right).$$

Therefore, (4.2) has two series of eigenvalues with $\mathrm{Re}\lambda_n>0$. As (4.3) shows, attenuation at infinity will hence occur more slowly than in the isotropic case. As in the anisotropic case also, exponentiality of the attenuation is conserved for the strain potential energy of the material.

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